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Note

Stirling Numbers and Records

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The equivalence of two classical sums giving the Stirling numbers of first kind results from a joint law for records.

The absolute values $a(n, k) = |s(n, k)|$ of the Stirling numbers of the first kind, $n \in \mathbb{N} = \{1, 2,..., \}$, $k = 1,..., n$, occur in probability due to one or the other of their expressions

$$a(n, k) = (n-1)! \sum{}' (t_1 \cdots t_{k-1})^{-1} = n! \sum{}'' \left[\prod_{j=1}^{q} (m_j! r_j^{m_j}) \right]^{-1}. \tag{1}$$

Here, $\sum'$ is over all sets of integers $0 < t_1 < \cdots < t_{k-1} < n$ and $\sum''$ over all sets of positive integers q, $r_1 < \cdots < r_q$, $m_1,..., m_q$ with $m_1 + \cdots + m_q = k$ and $r_1 m_1 + \cdots + r_q m_q = n$. For example, $\sum'$ occurs naturally in [3], and $\sum''$ in [2]. The equalities (1) result from the two expansions

$$\begin{aligned} \sum_{k=1}^{n} a(n, k)\, x^k &= x(x+1) \cdots (x+n-1), \\ \sum_{n \geqslant k} \frac{k!}{n!} a(n, k)\, x^n &= [-\ln(1-x)]^k. \end{aligned} \tag{2}$$

Standard texts contain a variety of proofs. Good and Tideman [1] give an 1826 reference. We give a justification based on the fact that

$$p(i_1,..., i_k; y_1,..., y_{k+1}) = \prod_{s=1}^{k} y_s^{i_s - 1} \tag{3}$$

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is the frequency-density function of a probability measure over $\mathbb{N}^k \times D_{k+1}$, with D_{k+1} the domain $0 < y_1 < \cdots < y_{k+1} < 1$. Where it occurs is indicated below. Integrating out sucessively $y_1,..., y_k$ and setting $i_1 + \cdots + i_s = t_s$, gives the marginal

$$p(i_1,..., i_k; y_{k+1}) = y_{k+1}^{t_k}(t_1 \cdots t_k)^{-1} \qquad \text{over} \quad \mathbb{N}^k \times [0, 1]. \tag{4}$$

Summation in $i_1,..., i_k$ then gives, keeping together the terms $t_k = n$ and using the first equality (1), the marginal density over $[0, 1]$,

$$p(y_{k+1}) = \sum_{n \geqslant k} a(n, k)\, y_{k+1}^n / n!. \tag{5}$$

On the other hand, summing first (3) in $i_1,..., i_k$ gives the marginal $p(y_1,..., y_{k+1}) = 1/(1-y_1) \cdots (1-y_k)$ over D_{k+1}, and integration in $y_1,..., y_k$ leaves

$$p(y_{k+1}) = [-\ln(1-y_{k+1})]^k / k!, \tag{6}$$

so comparison with (5) gives the second expansion (2), or the second equality (1).

For a sequence $X_1 X_2 \cdots$ of independent random variables with common distribution F having a density f, the interrecord times $T_k = L_{k+1} - L_k$, where $L_1 = 1$ and thereafter $L_k = \min\{j : X_j > X_{L_{k-1}}\}$, and the record values $Y_k = X_{L_k}$ have clearly the joint frequency-density function for $T_1,..., T_k$; $Y_1,..., Y_{k+1}$, at $(i_1,..., i_k) \in \mathbb{N}^k$, $y_1 < \cdots < y_{k+1}$:

$$p(i_1,..., i_k; y_1,..., y_{k+1}) = \prod_{s=1}^{k} [f(y_s) F^{i_s - 1}(y_s)]\, f(y_{k+1}). \tag{7}$$

Thus (3) is the case of X's with uniform law over $[0, 1]$. Starting with (7) instead of (3), the right-hand members (4) and (6) become, respectively,

$$F^{t_k}(y_{k+1}) f(y_{k+1})/(t_1 \cdots t_k), \quad \{-\ln[1 - F(y_{k+1})]\}^k f(y_{k+1})/k!.$$

Their quotient, summed over $0 < t_1 < \cdots < t_{k-1} < n = t_k$ gives for $0 < k \leqslant n$ the conditional law, where $z = F(y_{k+1})$,

$$P(L_{k+1} = n+1 \mid y_{k+1}) = (k!/n!)\, a(n, k)\, z^n [-\ln(1-z)]^{-k}.$$

This was obtained by Shorrock [4] in the case $f(t) = \exp(-t)$, $t \geqslant 0$, via a complicated Laplace transform argument.

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